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1992 J. Phys. A: Math. Gen. 25 5135

(http://iopscience.iop.org/0305-4470/25/19/021)

J. Phys. A: Math. Gen. **25** (1992) 5135-5140. Printed in the UK

The quantum group $SU_q(2)$ and the q-analogue of angular momenta in classical mechanics

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Received 7 November 1991, in final form 19 May 1992

Abstract. A set of classical quantities that obey the Poisson brackets corresponding to Lie commutation relations of the quantum group $SU_q(2)$ are obtained in classical mechanics. They are constructed in terms of a function $Q(l^2)$ and conventional angular momenta which satisfy the classical correspondence of the Lie commutation relation of the Lie algebra su(2). These quantities are called the q-analogue angular momenta, and their conservation law is mentioned. We give an example of $Q(l^2)$ that causes the classical q-analogue angular momenta to reduce to the conventional angular momenta while the quantum deformation parameter $\gamma = \ln q$ vanishes.

The quantum group $SU_q(2)$ (also denoted as $U_q(SU(2))$) plays an important role in statistical mechanics [1] and quantum integrable systems [2]. The quantum groups themselves are remarkable mathematical structures that emerged as algebraic abstractions from the inverse scattering problem [3] and conformal field theory [4]. In the last few years, interest in quantum groups and their applications in physics has grown substantially. A crucial aspect is the realization of quantum groups in physics models. Biedenharn, Macfarlane, Kulish, Damashinsky, Chaichian, Lukierski, Curtright, Zachos and Fairlie [5] have studied the q-boson realization and the so-called 'deforming functional' of $SU_q(2)$. Chen, Chang and Guo [6] gave the classical realization of $SU_q(2)$ via classical harmonic oscillators. It is of significance to research the structure of quantum groups in physics theory and to discuss the physical models that contain quantum groups.

In this paper we study $SU_q(2)$ from the point of view of classical mechanics. It is well known that the angular momenta in usual Newtonian mechanics obey the Poisson bracket relations that can be looked upon as the classical correspondence of the Lie bracket commutator relation of the Lie algebra su(2). Following this, when the $SU_q(2)$ commutator relations are given, one may find a set of Poisson brackets of physical quantities in three-dimensional Newtonian mechanics which is just the correspondence of the Lie bracket relation of $SU_q(2)$. We call these physical quantities the classical q-analogue angular momenta. In our discussion, the concrete physical model is not dealt with and the conclusion is universal. We also deal with the corresponding classical conversation law of the q-analogue angular momenta.

We start from the quantum group $SU_q(2)$ that is generated algebraically by the quantities J_+, J_- and J_z obeying the commutation relation

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = \frac{\sinh \gamma J_z}{\sinh \frac{1}{2}\gamma} \tag{2}$$

where we note

$$\gamma = \ln q \tag{3}$$

and q is a real number. As in the angular representation, we define

$$J_x = \tfrac{1}{2}(J_+ + J_-) \tag{4}$$

$$J_y = \frac{1}{2\mathrm{i}}(J_+ - J_-). \tag{5}$$

The commutator relations (1) and (2) can also be written as

$$[J_z, J_x] = \mathrm{i}J_y \tag{6}$$

$$[J_z, J_y] = -\mathrm{i}J_x \tag{7}$$

$$[J_x, J_y] = \frac{\mathrm{i}}{2}\frac{\sinh \gamma J_z}{\sinh \frac{1}{2}\gamma}. \tag{8}$$

Let us consider the mechanics of a particle moving in three-dimensional space. Its Hamiltonian can be expressed as

$$H = H(x, y, z, p_x, p_y, p_z) \tag{9}$$

where x, y and z coordinates of the particle in three-dimensional space, p_x, p_y and p_z are the corresponding momenta of the coordinates, and t labels the common time. We know that an arbitrary physical quantity s obeys the Jacobi equation [7]

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + \{s, H\} \tag{10}$$

in which the Poisson bracket is defined as

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial p_z} - \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial p_y}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial p_z}\frac{\partial g}{\partial z} \tag{11}$$

$$f = f(x, y, z, p_x, p_y, p_z, t) \qquad g = g(x, y, z, p_x, p_y, p_z, t).$$

In Newtonian mechanics, the angular momenta are specified in the following form:

$$l_x = yp_z - zp_y \qquad l_y = zp_x - xp_z \qquad l_z = xp_y - yp_x. \tag{12}$$

They satisfy

$$\{l_z, l_x\} = l_y \qquad \{l_z, l_y\} = -l_x \qquad \{l_x, l_y\} = l_z. \tag{13}$$

According to the principles of quantum mechanics [8], the correspondence between classical and quantum theory is given by

$$\{f, g\} \leftrightarrow \frac{1}{\mathrm{i}}[F, G] \tag{14}$$

where F and G are the operators in quantum mechanics corresponding to f and g, and the Planck constant is chosen as $\hbar = 1$. From (13) and (14) one has the commutation relations of the angular momentum operators L_x, L_y and L_z in quantum mechanics,

$$[L_z, L_x] = \mathrm{i}L_y \qquad [L_y, L_z] = \mathrm{i}L_x \qquad [L_x, L_y] = \mathrm{i}L_z$$

or

$$[L_z, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_z \tag{15}$$

where

$$L_{\pm} = L_x \pm \mathrm{i} L_y.$$

These commutation relations just determine the ordinary Lie algebra su(2). If we make use of (6)–(8) and (14), the classical correspondence of the commutation relations of the quantum group $\mathrm{SU}_q(2)$ can be deduced in the form

$$\{j_z, j_x\} = j_y \tag{16}$$

$$\{j_z, j_y\} = -j_x \tag{17}$$

$$\{j_x, j_y\} = \frac{1}{2} \frac{\sinh \gamma j_z}{\sinh \frac{1}{2}\gamma} \tag{18}$$

in which j_x, j_y and j_z are the classical corresponding quantities of the generators in $\mathrm{SU}_q(2)$. We call j_x, j_y and j_z the classical q-analogue angular momenta.

The purpose of our paper is to construct these classical q-analogue angular momenta j_x, j_y and j_z in terms of the kinetic parameters of a particle moving in the usual three-dimensional space. Comparing the Poisson bracket relations (16)–(18) with (13), we see that (16) and (17) are exactly the same as the former two equations of (13). Therefore, we can be assured that

$$j_z = l_z. \tag{19}$$

Using (11), (12), (16), (17) and (19) we obtain

$$p_y \frac{\partial j_x}{\partial p_x} - p_x \frac{\partial j_x}{\partial p_y} + y \frac{\partial j_x}{\partial x} - x \frac{\partial j_x}{\partial y} = j_y \tag{20}$$

$$p_y \frac{\partial j_y}{\partial p_x} - p_x \frac{\partial j_y}{\partial p_y} + y \frac{\partial j_y}{\partial x} - x \frac{\partial j_y}{\partial y} = -j_x. \tag{21}$$

Let Φ be a function of (x, y, z, p_x, p_y, p_z), which satisfies the vanishing Poisson bracket relation

$$\{l_z, \Phi\} = 0 \tag{22}$$

then it is easy to prove that when l_x and l_y satisfy (13),

$$j_x = l_x \Phi \qquad j_y = l_y \Phi \tag{23}$$

also satisfy the Poisson bracket relations (20) and (21). In the following we will discuss what expressions of function Φ makes j_x and j_y defined by (23) precisely satisfy the classical q-analogue Poisson bracket relation (18).

Using the well known relation

$$\{l_z, l_x^2 + l_y^2\} = 0$$

and from (22) we know that function Φ should be chosen as a function of l_z and $l_x^2 + l_y^2$ only, i.e.

$$\Phi = \Phi(X, Y) \tag{24}$$

where

$$X = l_z \qquad Y = l_x^2 + l_y^2. \tag{25}$$

Substituting (23) into (18), we have the partial differential equation

$$Y\left(2X\Phi\frac{\partial\Phi}{\partial Y}-\Phi\frac{\partial\Phi}{\partial X}\right)+X\Phi^2=\frac{1}{2}\frac{\sinh\gamma X}{\sinh\frac{1}{2}\gamma}. \tag{26}$$

Let us introduce

$$Z=\Phi^2. \tag{27}$$

Equation (25) then becomes

$$-\frac{Y}{2}\frac{\partial Z}{\partial X}+XY\frac{\partial Z}{\partial Y}+XZ=\frac{1}{2}\frac{\sinh\gamma X}{\sinh\frac{1}{2}\gamma}. \tag{28}$$

The general solution of (27) can be expressed as follows [9]:

$$Z=\frac{1}{Y}\left(Q(X^2+Y)-\frac{1}{\gamma}\frac{\cosh\gamma X}{\sinh\frac{1}{2}\gamma}\right) \tag{29}$$

where Q is an arbitrary differentiable function of argument $(Y+X^2)$. From (12) and (25) we have

$$l^2=Y+X^2=l_x^2+l_y^2+l_z^2. \tag{30}$$

Using (19), (23), (27), (29) and (30), the classical q-analogue angular momenta read

$$\begin{aligned}
j_x&=l_x(l_x^2+l_y^2)^{-1/2}\left(Q(l^2)-\frac{1}{\gamma}\frac{\cosh\gamma l_z}{\sinh\frac{1}{2}\gamma}\right)^{1/2}\\
j_y&=l_y(l_x^2+l_y^2)^{-1/2}\left(Q(l^2)-\frac{1}{\gamma}\frac{\cosh\gamma l_z}{\sinh\frac{1}{2}\gamma}\right)^{1/2}\\
j_z&=l_z.
\end{aligned} \tag{31}$$

It is easily verified that j_x, j_y and j_z obey (16)-(18) identically. From (31) we see that there is an arbitrary differentiable function $Q(l^2)$ in this expression. Choosing different $Q(l^2)$, one can define formulae of the classical q-analogue angular momenta. In an inverse procedure, when one quantizes the classical system with j_x, j_y and j_z defined by (31) and the Poisson bracket relations (16)-(18), the quantum q-analogue angular momenta with the Lie bracket relations obtained from (15), i.e. (6)-(8), realize the quantum group $SU_q(2)$.

Let us now deal with a simple case. Since $SU_q(2)$ contracts to the Lie algebra su(2) in the limit $q\to 1$ (i.e. $\gamma\to 0$), the classical q-analogue angular momenta should become the conventional angular momenta. This condition can be expressed as

$$\lim_{\gamma\to 0}\Phi=1. \tag{32}$$

Using (32), it is easy to see that the function $Q(l^2)$ can be chosen as

$$Q(l^2)=\frac{\cosh\gamma l}{\gamma\sinh\frac{1}{2}\gamma} \tag{33}$$

and the classical q-analogue angular momenta should take the following form:

$$\begin{aligned}
j_x&=\frac{l_x}{\sqrt{l_x^2+l_y^2}}\left(\frac{\cosh\gamma l-\cosh\gamma l_z}{\gamma\sinh\frac{1}{2}\gamma}\right)^{1/2}\\
j_y&=\frac{l_y}{\sqrt{l_x^2+l_y^2}}\left(\frac{\cosh\gamma l-\cosh\gamma l_z}{\gamma\sinh\frac{1}{2}\gamma}\right)^{1/2}\qquad j_z=l_z.
\end{aligned} \tag{34}$$

This is just the classical limit ($h \to 0$ with fixed γ) of the quantum expressions for the $SU_q(2)$ generators given by Babujian and Tsevelik [10].

Also of interest is in what system the classical q-analogue angular momenta are conserved. For these q-analogue angular momenta $\boldsymbol{j}$ this knowledge is necessary. First, although $\boldsymbol{j}$ has been expressed as a function of $\boldsymbol{l}$, and thus we know that $\boldsymbol{j}$ will be conserved whenever $\boldsymbol{l}$ is conserved, we do not know whether there is another case where $\boldsymbol{j}$ will be conserved when $\boldsymbol{l}$ is not conserved. Secondly, we also want to know in what condition each component of $\boldsymbol{l}$ is conserved. Since these q-analogue angular momenta are independent of time t, from the Jacobi equation (10) we deduce that in the case of

$$\{j_x, H\} = \{j_y, H\} = \{j_z, H\} = 0 \tag{35}$$

they are conserved.

Generally speaking, the Hamiltonian of a particle moving in three-dimensional space can be written as follows:

$$H = \frac{1}{m} \boldsymbol{p}^2 + V(x, y, z)$$

where m stands for the mass of the particle and $V(x, y, z)$ is an arbitrary potential function. In this case (35) become

$$\begin{aligned} l_x(l_x^2+l_y^2)\bigg\{2Q'\bigg[& l_x\left(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y}\right) + l_y\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right) + l_z\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right)\bigg] \\ &+ \frac{\sinh \gamma l_z}{\sinh \frac{1}{2}\gamma}\left(y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y}\right)\bigg\} - 2\left(Q - \frac{1}{\gamma}\frac{\cosh \gamma l_z}{\cosh \frac{1}{2}\gamma}\right) \\ &\times \left[l_y^2\left(z\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial z}\right) + l_x l_y\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right)\right] = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} l_y\left(l_x^2+l_y^2\right)\bigg\{2Q'\bigg[& l_x\left(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y}\right) + l_y\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right) + l_z\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right)\bigg] \\ &+ \frac{\sinh \gamma l_z}{\sinh \frac{1}{2}\gamma}\left(y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y}\right)\bigg\} - 2\left(Q - \frac{1}{\gamma}\frac{\cosh \gamma l_z}{\sinh \frac{1}{2}\gamma}\right) \\ &\times \left[l_x^2\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right) + l_x l_y\left(z\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial z}\right)\right] = 0 \end{aligned} \tag{37}$$

$$y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} = 0 \tag{38}$$

where

$$Q' = \frac{\mathrm{d}Q}{\mathrm{d}(l^2)}.$$

From the above formulae we see that the conservation conditions of the classical q-analogue angular momenta are not all the same as those of the conventional angular momenta. This highlights the difference between conventional q-analogue and angular momenta. Of course, this difference also means that there is disparity between the quantum group $SU_q(2)$ and the Lie algebra su(2).

However, if the potential function $V(x, y, z)$ is a central symmetry field, the classical q analogue angular momenta (31) are conserved naturally. Except for this case, only the potential function $V(x, y, z)$ cannot determine whether j is conserved. We note that for a given potential function $V(x, y, z)$ the properties of function $Q(l^2)$ and $\boldsymbol{l}$ play an important role in showing whether the classical q-analogue angular momenta j_x, j_y and j_z are conserved.

In this paper we have found three classical mechanics quantities j_x, j_y and j_z that satisfy the Poisson bracket relations corresponding to the Lie bracket relations of the quantum group $SU_q(2)$. These classical mechanics quantities are given by (31), and are called classical q-analogue angular momenta, and are determined by the differentiable function $Q(l^2)$ and the conventional three-dimensional angular momenta l_x, l_y and l_z. As an example, we determined a fixed $Q(l^2)$ that made the classical q-analogue angular momenta (34) reduce to the conventional angular momenta while the quantum deformation parameter $\gamma = \ln q$ vanished. We also gave the conservation condition of the above-mentioned classical q-analogue angular momenta, which is expressed as (36)–(38). Generally speaking, whether $\boldsymbol{j}$ is conserved is determined not only by the potential function but also by $Q(l^2)$ and the angular momenta $\boldsymbol{l}$.

When one quantizes the q-analogue angular momenta, the realization of the quantum group $SU_q(2)$ will be obtained in quantum mechanics. This is an interesting topic that will be discussed elsewhere.

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